

OR 6205: Deterministic Operations Research
Spring 2026

Deterministic Operations Research

Module 1: Linear Programming

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You should by the end of this class be able to:

- Recognize if the assumptions of LP apply to a given problem
- Distinguish between parameters and decision variables in a LP
- Explain how models are a product of context and decisions.
- Indicate assumptions made during formulation
- Evaluate whether a model adequately represents the intended situation

Recap

OR is an entire field devoted to understanding how to make better **decisions** using **mathematical models**.

It translates to billion of dollars in profit and millions of lives saved every year in different fields that goes from

Manufacturing (Sequencing, Batches, Queueing, work schedule)

Airport management (Crew schedule, airplanes x gates timetables)

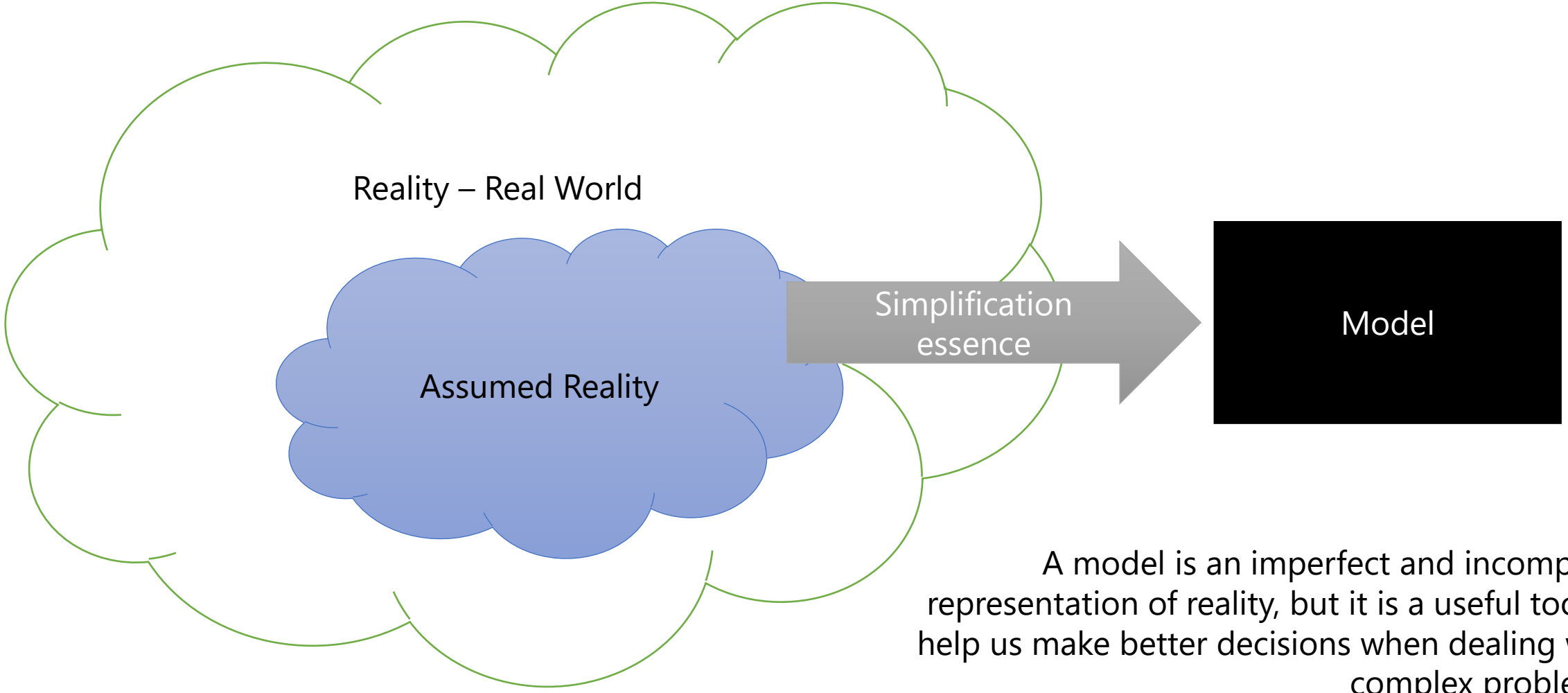
Food logistics and supply chain (distribution, location warehouses)

Health care (hospitals beds, EMTs location/dispatch)

Disaster management (Distribution supplies in at risk zones)

Social service (Shelter – how many beds?)

Recap



A model is an imperfect and incomplete representation of reality, but it is a useful tool to help us make better decisions when dealing with complex problems.

Recap: OR Framework



Data collection
Talk to stakeholders
Understand the mission

Transform essence of problem into set of equations

Transform essence of problem into set of equations

Incremental changes to improve the realism of the problem.
Validation
(Does the results make sense?)

Problems are not only technical, the are human.
Training and adjustments might be necessary to make the model be useful

Recap: OR Framework

Understand Problem

Data collection
Talk to stakeholders
Understand the mission

Formulate

Transform essence of Problem into set of equations

Solve

Transform essence of Problem into set of equations

Refine

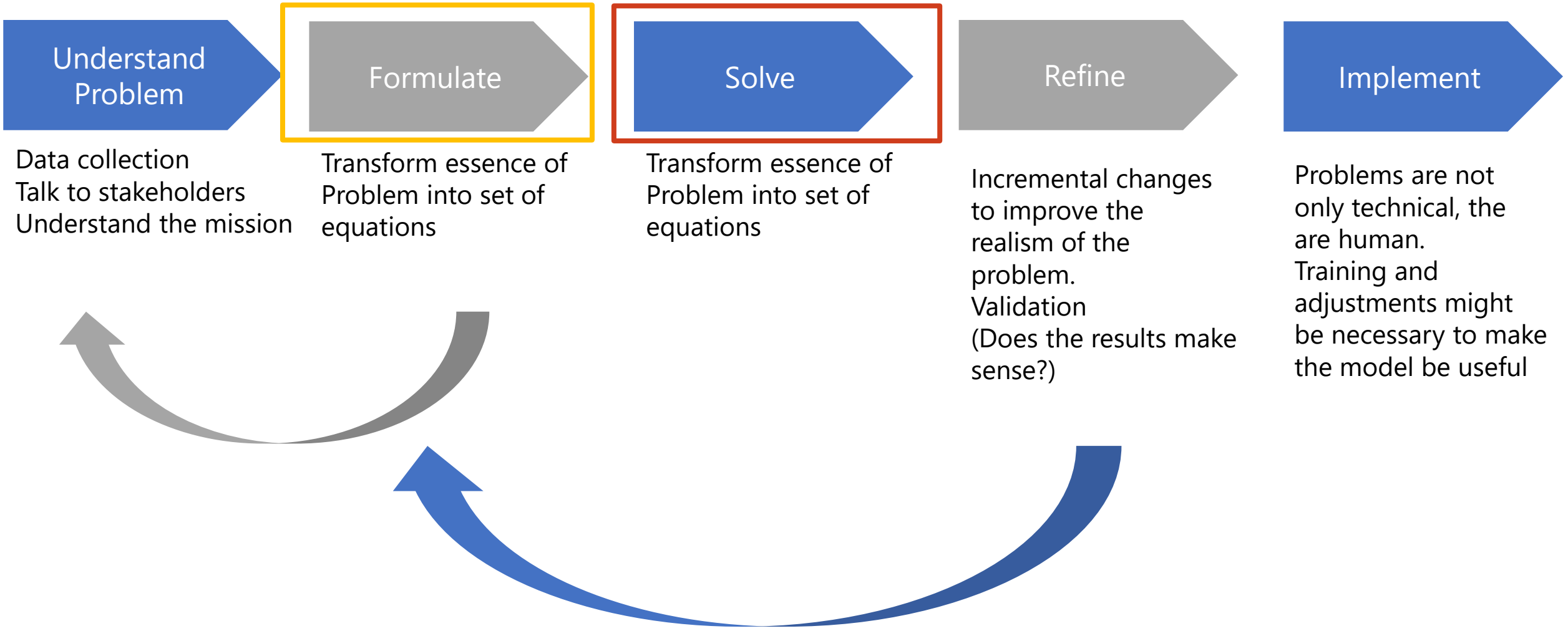
Incremental changes to improve the realism of the problem.
Validation
(Does the results make sense?)

Implement

Problems are not only technical, the are human.
Training and adjustments might be necessary to make the model be useful



Recap: OR Framework



LP

LP is a type of OR problem that uses a set of linear equations to represent the essence of the problem we are trying to make a decision.

LP was the first OR area, developed in the 1940's

It can solve a variety of problems, and provide the best decision, the optimal, given a set of parameters, data.

The most common application of LP is the allocation of limited resources **to competing activities** in the best possible (optimal) way. For example,

- Allocating production facilities to products
- Allocating limited funds to investment opportunities
- Selection of shipping patterns
- Agricultural planning

Remember the problem from last class?

$Max_x\ 3x_1 + 2x_2$

$2x_1 + x_2 \leq 100$

$x_1 + x_2 \leq 80$

$x_2 \leq 40$

$x_1, x_2 \geq 0$

What if the problem was different?

$$Max_x\ 3x_1$$

$$2x_1 \leq 100$$

$$x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

Would it be worth it to work on this model? Why?

Activity

Let’s start thinking about your problems

OR Terminology – Decision Variables

The variables that we optimize over are the **decision variables**.

They may be **continuous variables**, meaning that they can take on values in the real numbers, including fractions, or they may be **discrete variables**that can only take on integer values.

Continuous	Discrete
Amount of raw material to mix	Number of trucks to dispatch
Investment amount in dollars	Number of workers to schedule

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Continuous	Discrete
Amount of raw material to mix	Number of trucks to dispatch
Investment amount in dollars	Number of workers to schedule

x_1: Batch size (Number) of tables to make in the period
x_2: Batch size (Number) of chairs to make in the period

OR Terminology – Objective Function

The stakeholder goal, what they want to optimize.

The function being optimized is the **objective function.**

OR Terminology – Objective Function

The stakeholder goal, what they want to optimize.

The function being optimized is the **objective function.**

MAXIMIZE PROFIT: $Max_x\ 3x_1 + 2x_2$

OR Terminology - Constraints

The **domain over which this function is optimized** is the set of points that satisfy the constraints, which can be equations or inequalities.

We distinguish two kinds of constraints:

- **Functional constraints** – can have any form – relation between variables

- **Variable bounds** and Non-negative constraints – delimit values of each variable

OR Terminology - Constraints

The **domain over which this function is optimized** is the set of points that satisfy the constraints, which can be equations or inequalities.

We distinguish two kinds of constraints:

- **Functional constraints** – can have any form – relation between variables

Wood Limitation: $2x_1 + x_2 \leq 100$

Labor/hour limitation: x_1

- **Variable bounds** and Non-negative constraints – delimit values of each variable

Demand limitation: $x_1 \leq 40$

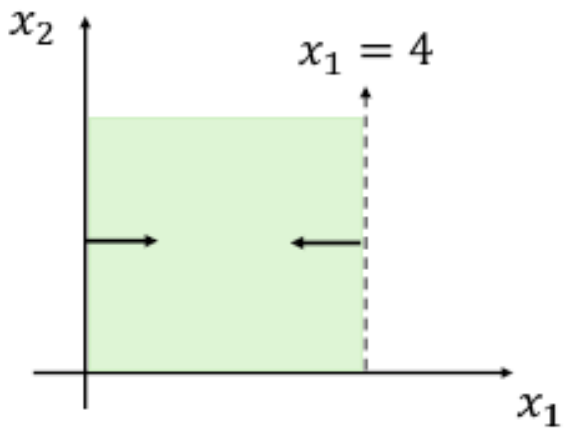
Non-negativity limitation: x_1, x_2

OR Terminology – Solutions: Feasible or Infeasible

- A **solution** to a mathematical program is any value **x** of the variables.

Consider

$$\max_{x_1} \{2x_1 \in R \mid x_1 < 4, x_1 \geq 0, x_2 \geq 0\}$$



A solution to this could be:

$$x_1 = 5$$

OR

$$x_1 = -18{,}439$$

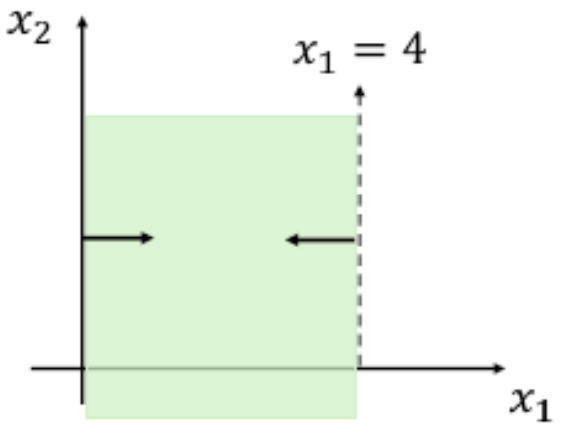
Would all solutions be allowed within the limits of the constraints?

OR Terminology – Solutions: Feasible or Infeasible

A **feasible solution** is a solution that satisfies all constraints, i.e. $\mathbf{x} \in S$.

Consider

$$\max_{x_1} \{2x_1 \in R \mid x_1 < 4, x_1 \geq 0, x_2 \geq 0\}$$



A feasible solution to this could be:

$$x_1 = 3$$

OR

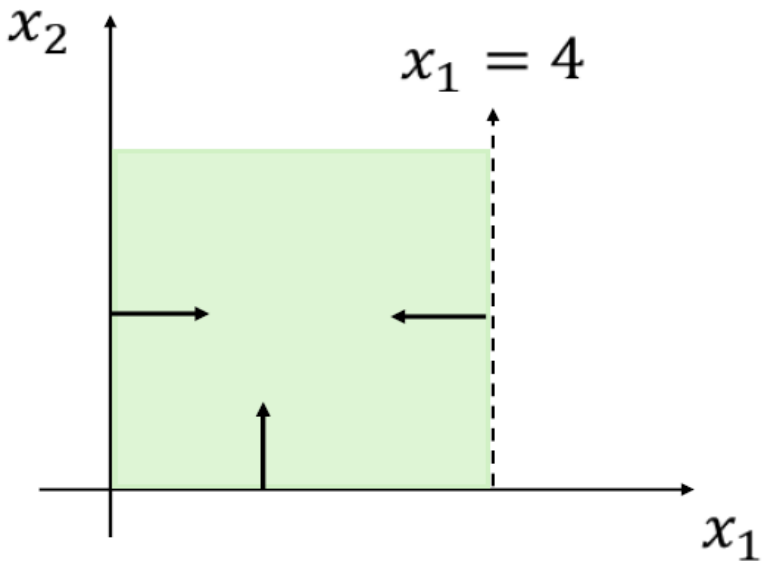
$$x_1 = 1$$

OR Terminology – Solutions: Feasible or Infeasible

We call set of all possible solutions or points that satisfy a given set of constraints the **feasible region**

Consider

$$\max_{x_1} \{2x_1 \in R \mid x_1 < 4, x_1 \geq 0, x_2 \geq 0\}$$

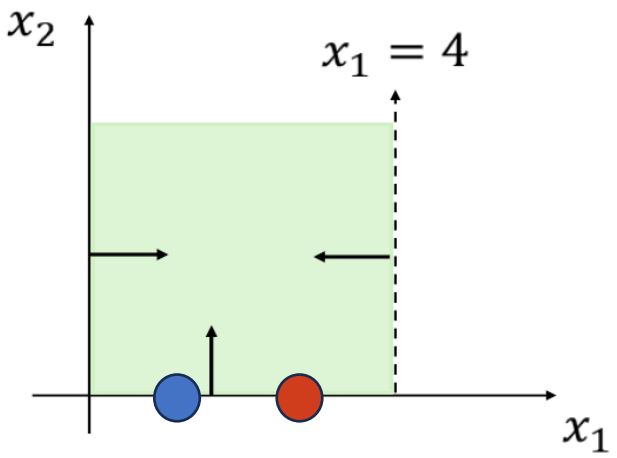


OR Terminology – Solutions: Feasible or Infeasible

The **value** of a solution is its objective function value.

Consider

$$\max_{x_1} \{2x_1 \in R \mid x_1 < 4, x_1 \geq 0, x_2 \geq 0\}$$



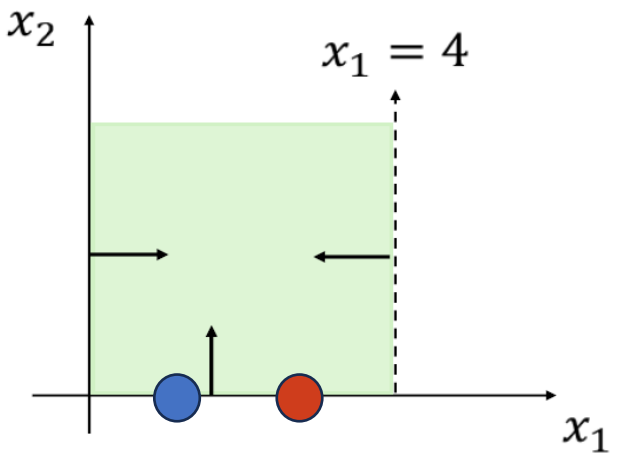
When $x_1 = 3$,,
the value of a solution would be ?
And when $x_1 = 1$,
the value of a solution would be ?

OR Terminology – Solutions: Feasible or Infeasible

The **value** of a solution is its objective function value.

Consider

$$\max_{x_1} \{2x_1 \in R \mid x_1 < 4, x_1 \geq 0, x_2 \geq 0\}$$



When $x_1 = 3$,,
the value of a solution would be 6
And when $x_1 = 1$,
the value of a solution would be 2

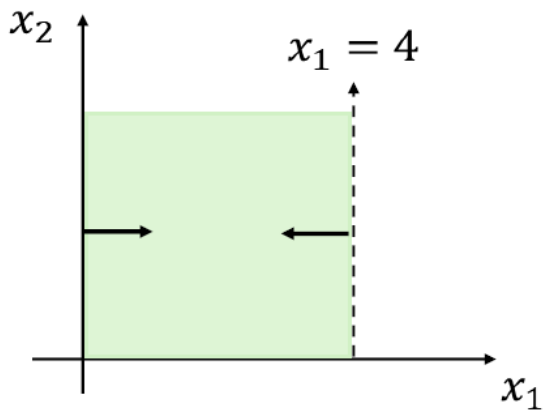
If x_2 =10?
And if objective function is $Z = 2x_2$?

OR Terminology – Solutions: Feasible or Infeasible

An **optimal solution** is a feasible solution that has a value at least as large (if maximizing) as any other feasible solution.

Consider an optimization problem

$$\max_{x_1} \{2x_1 \in R \mid x_1 < 4, x_1 \geq 0, x_2 > 0\}$$



The optimal solution to this problem would be ?

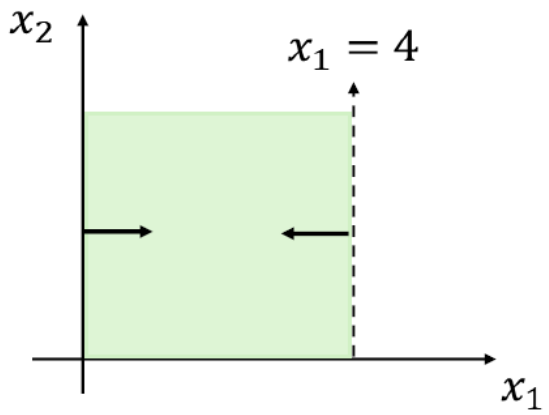
When at least one point or solution satisfies all the constraints in a problem, that problem has a **feasible solution**

OR Terminology – Solutions: Feasible or Infeasible

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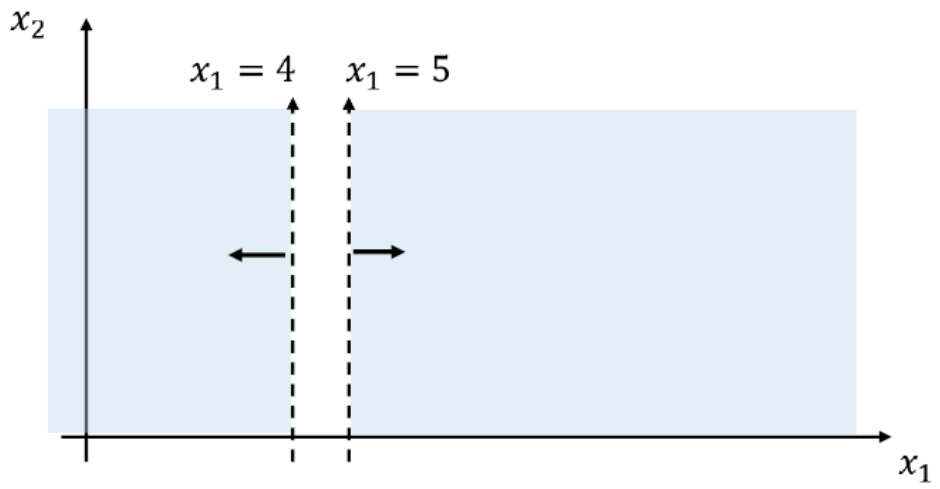


The optimal solution to this problem would be $x_1 = 3.\overline{99}$

When at least one point or solution satisfies all the constraints in a problem, that problem has a **feasible solution**

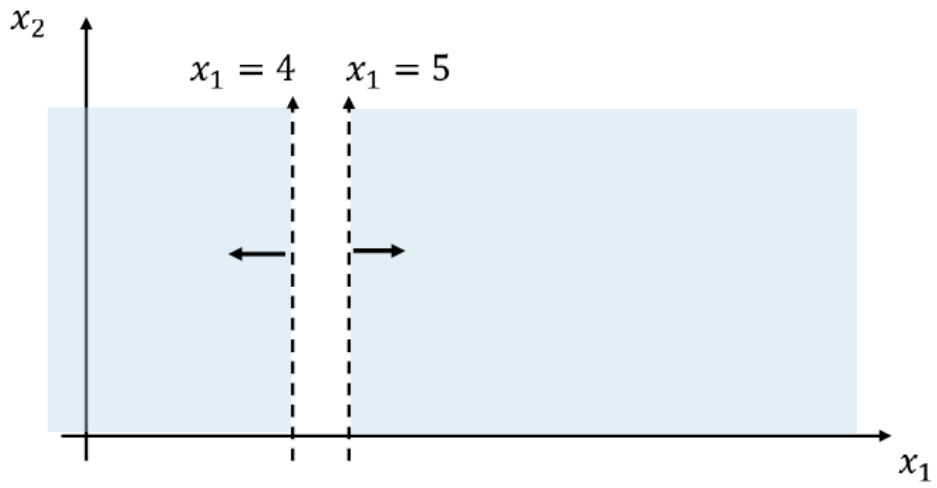
OR Terminology – Solutions: Feasible or Infeasible

Consider $\{x_1 \geq 5,\ x_1 \leq 4\}$. How does the feasible region look?



OR Terminology – Solutions: Feasible or Infeasible

Consider $\{x_1 \geq 5,\ x_1 \leq 4\}$. How does the feasible region look?



Infeasible problem

There are no solutions to this problem!

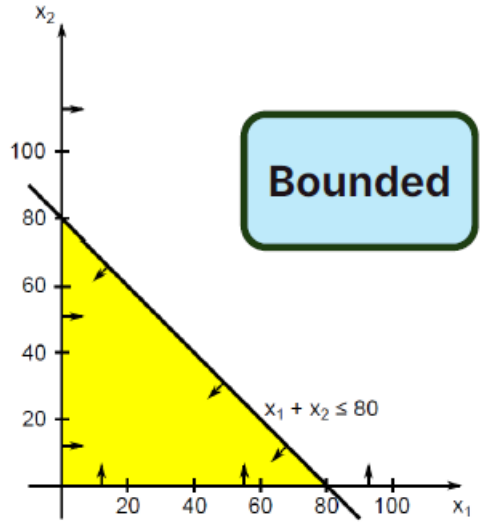
OR Terminology – Bounded or Unbounded

There are two different types of feasible regions:

bounded and unbounded

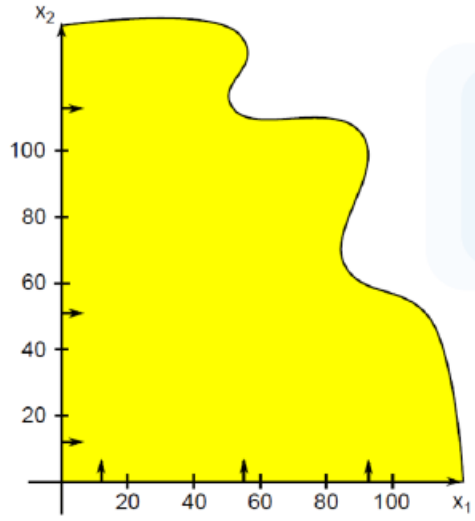
Consider:

- $\{x_1 + x_2 \leq 80, x_1 \geq 0, x_2 \geq 0\}$



Bounded

- $\{x_1 \geq 0,\ x_2 \geq 0\}$



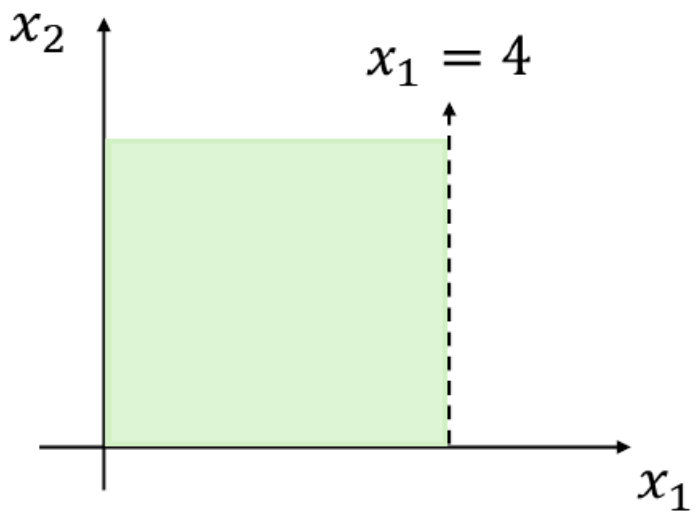
Unbounded

OR Terminology – Bounded or Unbounded

The problem is an **unbounded problem**, if for a maximization, there is a feasible solution **x** with value *f* (**x**) > *k* for every *k*

Consider an optimization problem

$$\max_{x_2} \{2x_2 \in R \mid x_1 < 4, x_1 \geq 0, x_2 \geq 0\}$$



When k=0, f(k) =0
When k=1, f(k) =2
When k=4, f(k)=8,
When k=1,000,000,000 f(k) = 2,000,000,000,
...

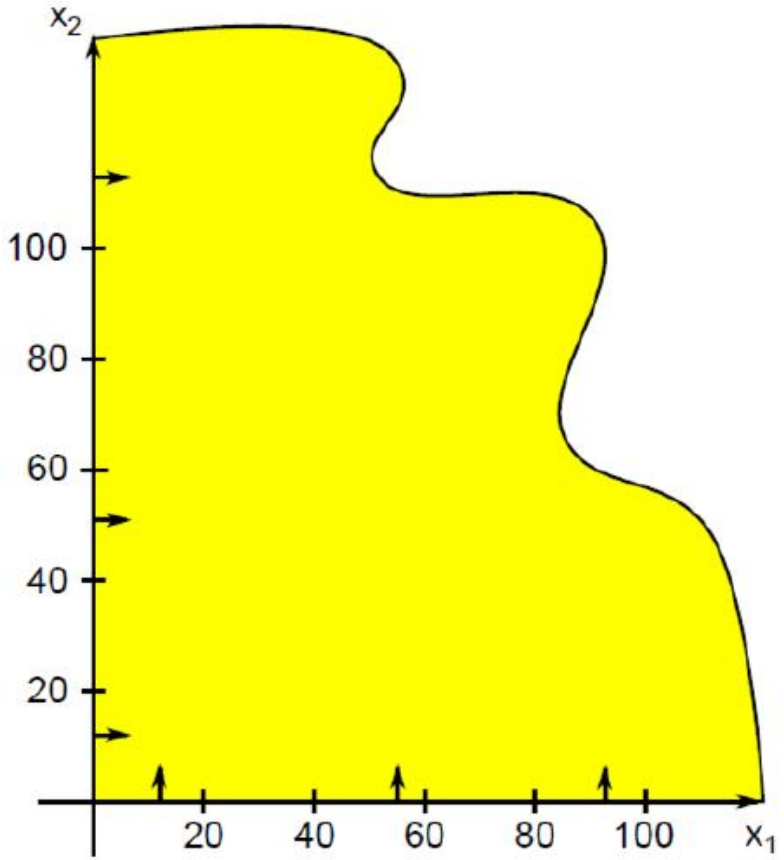
Unbounded problem

OR Terminology – Bounded or Unbounded

A problem with an unbounded feasible region CAN still have an optimal solution for certain objective functions.

→ How about

$$\min_{\{x_1,x_2\}} \{x_1 + x_2 |\ x_1 \geq 0,\ x_2 \geq 0\}$$



Linear Programming (LP)

LP uses a mathematical model of the system.

The name linear programming stands for the linearity of the functions used (linear), and the planning of activities (programming)

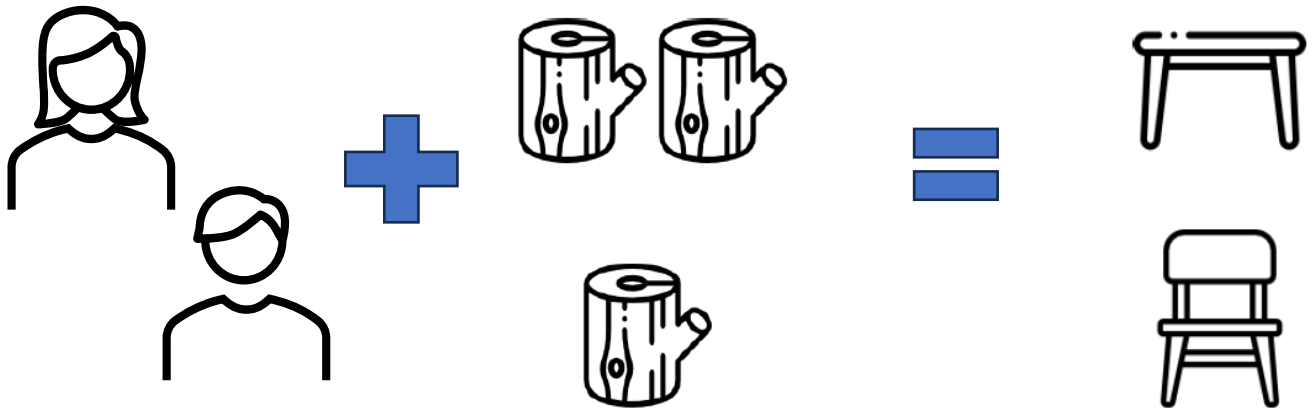
A linear programming (LP) problem is an optimization problem where:

We try to maximize (or minimize) a **linear function** of **continuous decision variables**.

Activity levels are represented by variables which are used to define the measure of performance (**objective function**), and restrictions of the problem are represented by functions in equality or inequality forms (**constraints**)

The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear inequality. Decision variables are allowed to take fractional values.

Example - Manufacturing



$$\begin{aligned} \text{maximize}_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 100 \quad \text{Wood} \\ & x_1 + x_2 \leq 80 \quad \text{Labor} \\ & x_1 \leq 40 \quad \text{Demand} \\ & x_1, x_2 \geq 0 \end{aligned}$$

Remember the Manufacturing example? Considering the definition of a **Linear Program**:

Do you think this **production problem** is a linear program?

Example – Manufacturing: Is it an LP?

$$\begin{aligned} \text{maximize}_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

✓ **Yes!**

Both the objective and constraints are linear

Example – Manufacturing: Is it an LP?

What if I was to say we cannot produce tables and chairs in fragments?

(e.g., we cannot produce 4.2 chairs nor 5.4 tables)

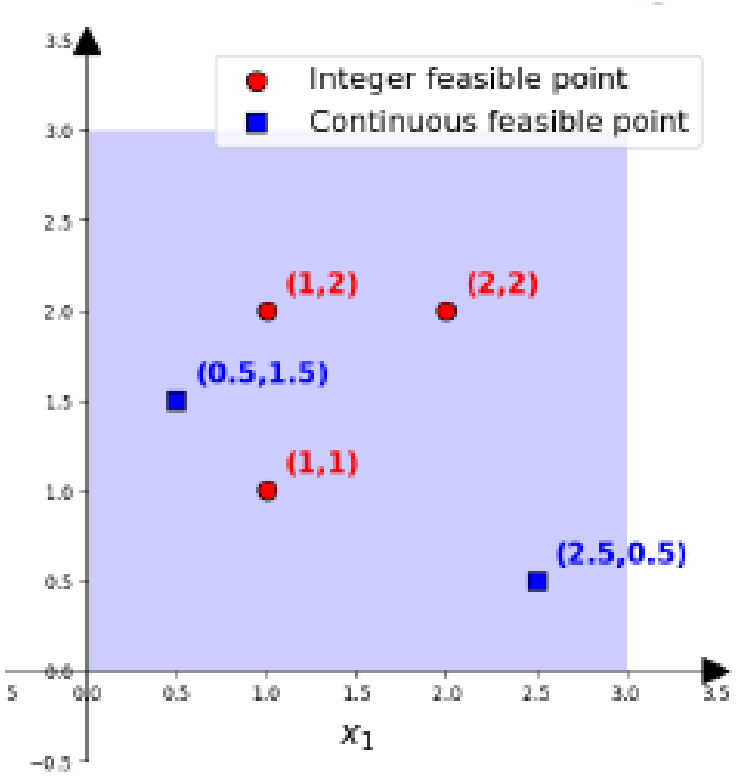
Example – Manufacturing: Is it an LP?

$$
\begin{aligned}
maximize_{x_1,x_2} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 100 \\
& x_1 + x_2 \leq 80 \\
& x_1 \leq 40 \\
& x_1, x_2 \geq 0, integer \\
& x_1, x_2 \in Z
\end{aligned}
$$

✓ Then the decision variables only can take integer values, and this will be an integer program

The variables are not continuous anymore, so, the feasible space cannot be represented by linear planes only.

Example MILP:



Example – Manufacturing: Is it an LP?

What if my decision variables are whether to produce tables or chairs?

Example – Manufacturing: Is it an LP?

$$\begin{aligned} \text{maximize}_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \in \{0,1\} \end{aligned}$$

✓ Then this will become a BP

The variables are now 0 or 1, binary, whether we produce one or the other.

When variables represent logical choices, they are usually defined so that

x = 1 if true (decide to take the action) and
x = 0 if false(decide not to take the action).

A variable whose possible values are 0 and 1 is called a **binary variable**.

Example – Manufacturing: Is it an LP?

$$\begin{aligned} \text{maximize}_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \in \{0,1\} \end{aligned}$$

✓ Then this will become a BP

When variables represent logical choices, they are usually defined so that

x = 1 if true (decide to take the action) and
x = 0 if false(decide not to take the action).

A variable whose possible values are 0 and 1 is called a **binary variable**.

The variables are now 0 or 1, binary, whether we produce one or the other.

But wait a minute, would this be a good model?

Example – Manufacturing: Is it an LP?

What if the profit made from selling a table was the square of the number of tables produced?

Example – Manufacturing: Is it an LP?

$$\begin{aligned} \text{maximize}_{x_1,x_2} \quad & 3x_1^2 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \in \{0,1\} \end{aligned}$$

✓ Then this will become an NLP

There is now a nonlinear relation among the variable x_1 and profit. This model does not have only linear equations.

OR Fields

LP – Linear Programming

IP – Integer Programming

MILP – Mixed Integer-linear Programming

Binary IP – Binary Integer Programming

NLP – Nonlinear Programming

Network Optimization

Dynamic Programming

Queueing Theory

Inventory Theory

Forecasting

Constraint Optimizations

Stochastic Optimization (e.g.: Markov Chains, Markov Decision Process)

Simulation

Game Theory

OR Fields

LP – Linear Programming

NLP – Nonlinear Programming

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Network Optimization

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Multiple Criteria Decision Making

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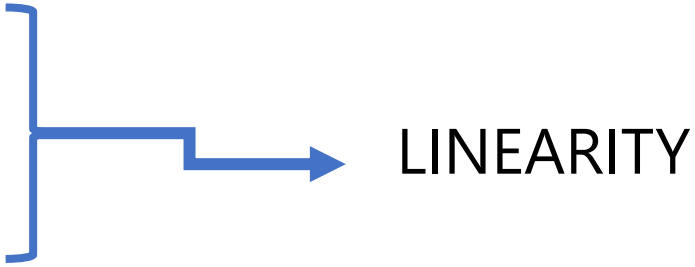
Assumptions ok for this model?

Proportionality

Additivity

Divisibility

Certainty



Assumptions of Linear Programming

Proportionality

$$\begin{aligned} \text{maximize}_{x_1,x_2} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The proportionality assumption says that the contribution to the objective function from each decision variable is proportional to the value of the decision variable.

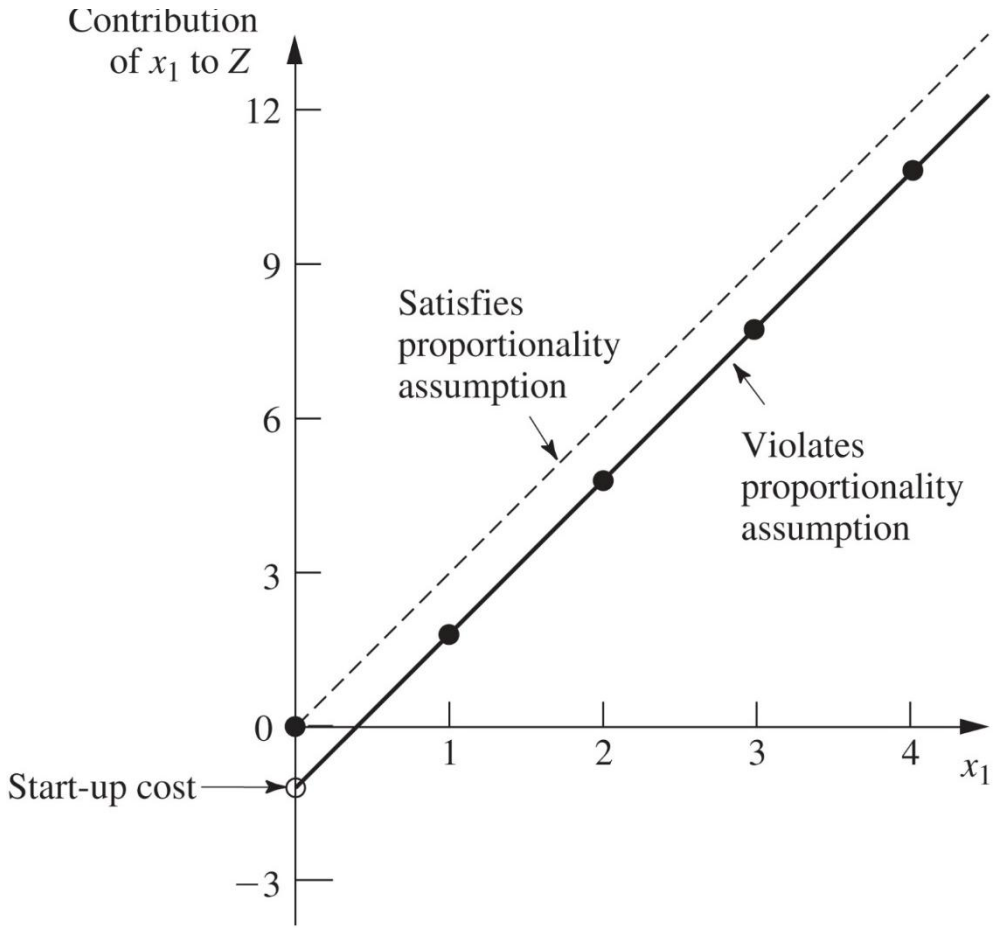
Contribution of each activity to the value of the objective function/LHS of each functional constraint is proportional to the level of the activity

If the assumption does not hold, one must use **nonlinear programming**

Assumptions of Linear Programming

Proportionality

x1	Profit from Product 1	
	Proportionality Satisfied	Proportionality Violated
0	0	0
1	3	2
2	6	5
3	9	8
4	12	11



Assumptions of Linear Programming

Additivity

$$
\begin{aligned}
maximize_{x_1,x_2} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 100 \\
& x_1 + x_2 \leq 80 \\
& x_1 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The additivity assumption states that the contribution to the objective function for any variable is independent of the other decision

Every function in a LP model is the sum of the individual contributions of the activities

Give me an example of an OF that would not respect this assumptions

Assumptions of Linear Programming

Divisibility

$$
\begin{aligned}
maximize_{x_1,x_2} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 100 \\
& x_1 + x_2 \leq 80 \\
& x_1 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The divisibility assumption says that each decision variable is permitted to assume fractional values

Decision variables in a LP model may have any values

- Including non-integer values

Assumes activities can be run at fractional values

Assumptions of Linear Programming

Certainty

$$
\begin{aligned}
\text{maximize}_{x_1,x_2} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 100 \\
& x_1 + x_2 \leq 80 \\
& x_1 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The certainty assumption says that all parameters (coefficients) are known with certainty (i.e., they are not variable).

Value assigned to each parameter of a LP model is assumed to be a known constant

Seldom satisfied precisely in real applications

- Sensitivity analysis used

Activity

Can you model your problem as a Linear Programming model?

- Proportionality
- Additivity
- Divisibility
- Certainty

Activity

Let’s formulate problems – what are the steps?

Activity

Let's formulate problems – what are the steps?

Step 1 : Clearly define the decision variables

Step 2 : Construct the objective function

Step 3 : Formulate the constraints

Example Problem: A Disaster Aid Problem

After a major disaster, an aid organization wants to quickly send tents and food using a cargo plane.

The cargo plane has a weight and size capacity, and they cannot carry all supplies needed or we don't have all supply available.

The total weight of all pallets cannot exceed 40,000 lbs.

There is space for at most 6 pallets on a plane

Only up to 5 pallets of food are currently available to send

The benefits for the people needing support were quantified based on most need. The benefits are 8 units per each pallet of tent and 6 units per each pallet of food sent

Example Problem: A Disaster Aid Problem

Table: Data for Sending Aid

	Tents	Food
Weight (1000 lbs)	7.5	5
Cost ($1000/pallet)	11	4
Criticality	5	8
Expected Benefit	8	6



How many pallets of tents and food must be sent to achieve the highest possible benefit?



Step 1: Define decision variables



How many pallets of tent should we send in a cargo plane?



How many pallets of food should we send in a cargo plane?



Step 2: Form objective function

What is our aim/objective?

To find the loading plan that will **maximize** the benefit

Benefit: 8  + 6 



Step 2: Form objective function

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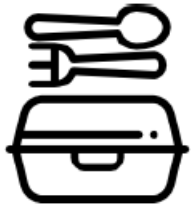
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Step 2: Form objective function

What is our aim/objective?

To find the loading plan that will **maximize** the benefit

Benefit: 8  + 6 

02

maximize 8  + 6 



Step 3: Add constraints

x →  produced each week

y →  produced each week



Step 3: Add constraints

x ⟶  produced each week

y ⟶  produced each week

Table: Data for Sending Aid

	Tents	Food
Weight (1000 lbs)	7.5	5
Cost ($1000/pallet)	11	4
Criticality	5	8
Expected Benefit	8	6



Step 3: Add constraints

x →  produced each week

y →  produced each week

03

subject to $7.5x + 5y \leq 40$ *Weight capacity*

$x + y \leq 6$ *Space limit*

$y \leq 5$ *Food available*

$x, y \geq 0$ *Non-negativity clause*



Step 4: Complete the problem

$$
\begin{aligned}
&\max && 8x + 6y && \text{Objective function}\\
&\text{s.t} && 7.5x + 5y <= 40 && \text{Constraints}\\
& && x + y <= 6 &&\\
& && y <= 5 &&\\
& && x,y >= 0 &&
\end{aligned}
$$

Let's solve it?



General Linear Programming Model

Objective

$$Maximize_x\ 8x_1 + 6x_2$$

Subject to:

Functional Constraints

$$7.5x + 5y \leq 40$$
$$x + y = 6$$
$$y \leq 5$$

Bounds

$$x, y \geq 0$$

A system with 2 variables and 3 constraints

where x *and* y are the decision variables.

$$Maximize_x\ c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \{\leq, \geq, =\} \quad b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \{\leq, \geq, =\} \quad b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \{\leq, \geq, =\} \quad b_m$$

$$0 \leq x_j, \quad j = 1, \ldots, n$$

A system with n variables and m constraints

where a_{ij}, b_j, c_j are the model parameters

and $x_1, x_2 \ldots x_n$ are the decision variables.

General Linear Programming Model

Objective

$$Maximize_{\boldsymbol{x}}\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$Maximize_{\boldsymbol{x}}\ Z = \sum_i c_ix_i$$

Objective

Subject to:

Functional Constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \{\leq, \geq, =\} \quad b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \{\leq, \geq, =\} \quad b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \{\leq, \geq, =\} \quad b_m$$

$$\sum_i a_ix_i \geq b_i - \forall\ i \in R_1$$
$$\sum_i a_ix_i = b_i - \forall\ i \in R_2$$
$$\sum_i a_ix_i = b_i - \forall\ i \in R_1$$

Functional Constraints

Bounds

$$0 \leq x_j, \quad j = 1, \ldots, n$$

$$x_i \geq 0\ \forall\ i$$

Bounds

A system with n variables and m constraints where a_{ij}, b_j, c_j are the model parameters and $x_1, x_2 \ldots x_n$ are the decision variables.

x_i = decision variables (i is the index of the set of DV, rep each DV)

c, a, b =parameters

General Linear Programming Model

Objective

$$Maximize_{\boldsymbol{x}}\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$Maximize_x\ Z = \sum_i c_ix_i$$

Or $Maximize_{\boldsymbol{x}}\ \boldsymbol{c^{-1}x}$

Subject to:

Functional Constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\ \{\leq, \geq, =\}\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\ \{\leq, \geq, =\}\ b_2 \\ \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\ \{\leq, \geq, =\}\ b_m \end{aligned}$$

$$\begin{aligned} \textstyle\sum_i a_ix_i &\geq b_i - \forall\ i \in R_1 \\ \textstyle\sum_i a_ix_i &= b_i - \forall\ i \in R_2 \\ \textstyle\sum_i a_ix_i &= b_i - \forall\ i \in R_1 \end{aligned}$$

or $A_1\boldsymbol{x} \geq B_1$
$A_2\boldsymbol{x} = B_2$
$A_3\boldsymbol{x} \leq B_3$

Bounds

$$0 \leq x_j, \quad j = 1, \ldots, n$$

$$x_i \geq 0\ \forall\ i$$

or $\boldsymbol{x} \geq 0$

A system with n variables and m constraints
where a_{ij}, b_j, c_j are the model parameters
and $x_1, x_2 \ldots x_n$ are the decision variables.